4727 Further Pure Mathematics 3

1 (i) (a)	$(n=)\ 3$	B1	**1**	For correct n
(b)	$(n=)\ 6$	B1	**1**	For correct n
(c)	$(n=)\ 4$	B1	**1**	For correct n
(ii)	$(n=)\ 4,\ 6$	B1		For *either* 4 *or* 6
		B1	**2**	For both 4 and 6 and no extras Ignore all $n \geqslant 8$ **SR** B0 B0 if more than 3 values given, even if they include 4 or 6
			$\boxed{5}$	
2 (i)	$\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}} \times \frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}+\mathrm{i}} = \frac{1}{2}+\frac{1}{2}\mathrm{i}\sqrt{3}$	M1		For multiplying top and bottom by complex conjugate
	OR $\frac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}} = \frac{2e^{\frac{1}{6}\pi\mathrm{i}}}{2e^{-\frac{1}{6}\pi\mathrm{i}}}$			*OR* for changing top and bottom to polar form
	$=(1)e^{\frac{1}{3}\pi\mathrm{i}}$	A1		For $(r=)\ 1$ (may be implied)
		A1	**3**	For $(\theta=)\ \frac{1}{3}\pi$ **SR** Award maximum A1 A0 if $e^{\mathrm{i}\theta}$ form is not seen
(ii)	$\left(e^{\frac{1}{3}\pi\mathrm{i}}\right)^6 = e^{2\pi\mathrm{i}} = 1 \Rightarrow \quad (n=)\ 6$	M1		For use of $e^{2\pi\mathrm{i}}=1$, $e^{\mathrm{i}\pi}=-1$, $\sin k\pi = 0$ *or* $\cos k\pi = \pm 1$ (may be implied)
		A1	**2**	For $(n=)\ 6$ **SR** For $(n=)\ 3$ only, award M1 A0
			$\boxed{5}$	
3 (i)	$\mathbf{n} = [2,1,3]\times[3,1,5]$	M1		For using direction vectors and attempt to find vector product
	$=[2,-1,-1]$	A1	**2**	For correct direction (allow multiples)
(ii)	$d = \frac{\lvert[5,2,1].[2,-1,-1]\rvert}{\sqrt{6}}$	B1		For $(\mathbf{AB}=)\ [5,2,1]$ or any vector joining lines
		M1		For attempt at evaluating $\mathbf{AB}.\mathbf{n}$
		M1		For $\lvert\mathbf{n}\rvert$ in denominator
	$=\frac{7}{\sqrt{6}} = \frac{7}{6}\sqrt{6} = 2.8577$	A1	**4**	For correct distance
			$\boxed{6}$	

4	$m^2+4m+5\,(=0) \Rightarrow m=\dfrac{-4\pm\sqrt{16-20}}{2}$	M1		For attempt to solve correct auxiliary equation
	$=-2\pm\mathrm{i}$	A1		For correct roots
	CF $=\mathrm{e}^{-2x}(C\cos x+D\sin x)$	A1√		For correct CF (here or later). f.t. from m **AEtrig** but not forms including $\mathrm{e}^{\mathrm{i}x}$
	PI $= p\sin 2x+q\cos 2x$	B1		For stating a trial PI of the correct form
	$y'=2p\cos 2x-2q\sin 2x$ $y''=-4p\sin 2x-4q\cos 2x$	M1		For differentiating PI twice and substituting into the DE
	$\cos 2x(-4q+8p+5q)$ $+\sin 2x(-4p-8q+5p) = 65\sin 2x$	A1		For correct equation
	$\left.\begin{array}{l}8p+q=0\\ p-8q=65\end{array}\right\}\quad p=1,\quad q=-8$	M1		For equating coefficients of $\cos 2x$ and $\sin 2x$ and attempting to solve for p and/or q
	PI $=\sin 2x-8\cos 2x$	A1		For correct p and q
	$\Rightarrow y=$ $\mathrm{e}^{-2x}(C\cos x+D\sin x)+\sin 2x-8\cos 2x$	B1√	9	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI
			$\boxed{9}$	
5 (i)	$y=u-\dfrac{1}{x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{1}{x^2}$	M1 A1		For differentiating substitution For correct expression
	$x^3\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}+\dfrac{1}{x^2}\right)=x\left(u-\dfrac{1}{x}\right)+x+1$	M1		For substituting y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ into DE
	$\Rightarrow x^2\dfrac{\mathrm{d}u}{\mathrm{d}x}=u$	A1	4	For obtaining correct equation **AG**
(ii)	METHOD 1			
	$\displaystyle\int\frac{1}{u}\,\mathrm{d}u=\int\frac{1}{x^2}\,\mathrm{d}x \Rightarrow \ln ku=-\frac{1}{x}$	M1 A1		For separating variables and attempt at integration For correct integration (k not required here)
	$ku=\mathrm{e}^{-1/x} \Rightarrow k\left(y+\dfrac{1}{x}\right)=\mathrm{e}^{-1/x}$	M1 M1		For any 2 of / For all 3 of $\left\}\begin{array}{l}k \text{ seen,}\\ \text{exponentiating,}\\ \text{substituting for } u\end{array}\right.$
	$\Rightarrow y=A\mathrm{e}^{-1/x}-\dfrac{1}{x}$	A1	5	For correct solution **AEF** in form $y=\mathrm{f}(x)$
	METHOD 2			
	$\dfrac{\mathrm{d}u}{\mathrm{d}x}-\dfrac{1}{x^2}u=0 \Rightarrow$ I.F. $\mathrm{e}^{\int -1/x^2\,\mathrm{d}x}=\mathrm{e}^{1/x}$	M1		For attempt to find I.F.
	$\Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}x}\left(u\,\mathrm{e}^{1/x}\right)=0$	A1		For correct result
	$u\,\mathrm{e}^{1/x}=k \Rightarrow y+\dfrac{1}{x}=k\,\mathrm{e}^{-1/x}$	M1 M1		From $\boxed{u\times\text{I.F.}=}$, $\left.\begin{array}{l}\text{for } k \text{ seen}\\ \text{for substituting for } u\end{array}\right\}$ in either order
	$\Rightarrow y=k\,\mathrm{e}^{-1/x}-\dfrac{1}{x}$	A1		For correct solution **AEF** in form $y=\mathrm{f}(x)$
			$\boxed{9}$	

6 (i)	METHOD 1 Use 2 of $[-4, 2, 0], [0, 0, 3], [-4, 2, 3], [4, -2, 3]$ or multiples	M1		For finding vector product of 2 appropriate vectors in plane *ACGE*
	$\mathbf{n} = k\,[1, 2, 0]$	A1		For correct **n**
	Use $A[4, 0, 0], C[0, 2, 0], G[0, 2, 3]$ *OR* $E[4, 0, 3]$	M1		For substituting a point in the plane
	$\mathbf{r}.[1, 2, 0] = 4$	A1	**4**	For correct equation. **AEF** in this form
	METHOD 2 $\mathbf{r} = [4, 0, 0] + \lambda[-4, 2, 0] + \mu[0, 0, 3]$	M1		For writing plane in 2-parameter form
	$\Rightarrow x = 4 - 4\lambda,\ y = 2\lambda,\ z = 3\mu$	A1		For 3 correct equations
	$x + 2y = 4$	M1		For eliminating λ (and μ)
	$\Rightarrow \mathbf{r}.[1, 2, 0] = 4$	A1		For correct equation. **AEF** in this form
(ii)	$\theta = \cos^{-1}\dfrac{\lvert[3, 0, -4].[1, 2, 0]\rvert}{\sqrt{3^2+0^2+4^2}\sqrt{1^2+2^2+0^2}}$	B1√ M1 M1		For using correct vectors (allow multiples). f.t. from **n** For using scalar product For multiplying both moduli in denominator
	$\theta = \cos^{-1}\dfrac{3}{5\sqrt{5}} = 74.4°$ $(74.435...°, 1.299...)$	A1	**4**	For correct angle
(iii)	*AM*: $(\mathbf{r} =)\,[4, 0, 0] + t[-2, 2, 3]$ $(or\ [2, 2, 3] + t[-2, 2, 3])$	M1 A1		For obtaining parametric expression for *AM* For correct expression seen or implied
	$3(4 - 2t) - 4(3t) = 0$ $(or\ 3(2 - 2t) - 4(3 + 3t) = 0)$	M1		For finding intersection of *AM* with *ACGE*
	$t = \frac{2}{3}\ (or\ t = -\frac{1}{3})$ *OR* $\mathbf{w} = \left[\frac{8}{3}, \frac{4}{3}, 2\right]$	A1		For correct t *OR* position vector
	$AW : WM = 2 : 1$	A1	**5**	For correct ratio
		$\boxed{13}$		
7 (i) (a)	$x + y - a \in \mathbb{R}$	B1		For stating closure is satisfied
	$(x * y) * z = (x + y - a) * z = x + y + z - 2a$	M1		For using 3 distinct elements bracketed both ways
	$x * (y * z) = x * (y + z - a) = x + y + z - 2a$	A1		For obtaining the same result twice for associativity **SR** 3 distinct elements bracketed once, expanded, and symmetry noted scores M1 A1
	$x + e - a = x \Rightarrow e = a$	B1		For stating identity = a
	$x + x^{-1} - a = a \Rightarrow x^{-1} = 2a - x$	M1 A1	 **6**	For attempting to obtain inverse of x For obtaining inverse $= 2a - x$ *OR* for showing that inverses exist, where $x + x^{-1} = 2a$
(b)	$x + y - a = y + x - a \Rightarrow$ commutative	B1	**1**	For stating commutativity is satisfied, with justification
(c)	x order 2 $\Rightarrow x * x = e \Rightarrow 2x - a = e$	M1		For obtaining equation for an element of order 2
	$\Rightarrow 2x - a = a \Rightarrow x = a = e$ *OR* $x = x^{-1} \Rightarrow x = 2a - x \Rightarrow x = a = e$ $\Rightarrow$ no elements of order 2	A1	**2**	For solving and showing that the only solution is the identity (which has order 1) *OR* For proving that there are no self-inverse elements (other than the identity)

(ii)	e.g. $2+1-5=-2 \notin \mathbb{R}^+$	M1		For attempting to disprove closure
	$\Rightarrow$ not closed	A1		For stating closure is not necessarily satisfied ($0<x+y \leqslant 5$ required)
	e.g. $2\times5-11=-1 \notin \mathbb{R}^+$	M1		For attempting to find an element with no inverse
	$\Rightarrow$ no inverse	A1	**4**	For stating inverse is not necessarily satisfied ($x \geqslant 10$ required)
		$\boxed{13}$		
8 (i)				z may be used for $e^{i\theta}$ throughout
	$\sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$	B1		For expression for $\sin\theta$ seen or implied
		M1		For expanding $\left(e^{i\theta}-e^{-i\theta}\right)^6$
	$\sin^6\theta=$			At least 4 terms and 3 binomial coefficients required.
	$-\frac{1}{64}\left(e^{6i\theta}-6e^{4i\theta}+15e^{2i\theta}-20+15e^{-2i\theta}-6e^{-4i\theta}+e^{-6i\theta}\right)$	A1		For correct expansion. Allow $\frac{\pm(i)}{64}\left(\cdots\cdots\right)$
	$=-\frac{1}{64}\left(2\cos6\theta-12\cos4\theta+30\cos2\theta-20\right)$	M1		For grouping terms and using multiple angles
	$\sin^6\theta=-\frac{1}{32}\left(\cos6\theta-6\cos4\theta+15\cos2\theta-10\right)$	A1	**5**	For answer obtained correctly **AG**
(ii)	$\cos^6\theta=$ *OR* $\sin^6\left(\frac{1}{2}\pi-\theta\right)=$	M1		For substituting $\left(\frac{1}{2}\pi-\theta\right)$ for θ throughout
	$-\frac{1}{32}\left(\cos(3\pi-6\theta)-6\cos(2\pi-4\theta)+15\cos(\pi-2\theta)-10\right)$	A1		For correct unsimplified expression
	$\cos^6\theta=\frac{1}{32}\left(\cos6\theta+6\cos4\theta+15\cos2\theta+10\right)$	A1	**3**	For correct expression with $\cos n\theta$ terms **AEF**
(iii)	$\int_0^{\frac{1}{4}\pi}\frac{1}{32}\left(-2\cos6\theta-30\cos2\theta\right)\,d\theta$	B1√		For correct integral. f.t. from $\sin^6\theta-\cos^6\theta$
	$=-\frac{1}{16}\left[\frac{1}{6}\sin6\theta+\frac{15}{2}\sin2\theta\right]_0^{\frac{1}{4}\pi}$	M1		For integrating $\cos n\theta, \sin n\theta$ *or* $e^{in\theta}$
		A1√		For correct integration. f.t. from integrand
	$=-\frac{11}{24}$	A1	**4**	For correct answer **WWW**
		$\boxed{12}$		